Segmentation of ARX-models Using Sum-of-Norms Regularization

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Segmentation of time-varying systems and signals into models whose parameters are piecewise constant in time is an important and well studied problem. It is here formulated as a least-squares problem with sum-of-norms regularization over the state parameter jumps, a generalization of ℓ_1-regularization. A nice property of the suggested formulation is that it only has one tuning parameter, the regularization constant which is used to trade off fit and the number of segments.

Model or signal segmentation is common in e.g. signal analysis (like speech and seismic data), failure detection and diagnosis. We here consider segmentation of ARX-models and hence seek a model

$$y(t) = \varphi^T(t)\theta(t). \tag{1}$$

where the system parameters are piecewise constant, and change only at certain time instants t_k that are more or less rare:

$$\theta(t) = \theta_k, \quad t_k < t \le t_{k+1}. \tag{2}$$

If we allow all the parameter values in (1) to be free in a least-squares criterion we would get a perfect fit, at the price of models that adjust in every time step, following any momentary noise influence. Such a grossly over-fit model would have no generalization ability, and so would not be very useful.

Sum-of-Norms Regularization

To penalize model parameter changes over time, we add a penalty or regularization term that is a sum of norms of the parameter changes:

$$\min_{\theta(t)} \sum_{t=1}^{N} \|y(t) - \varphi^T(t)\theta(t)\|^2 + \lambda \sum_{t=2}^{N} \|\theta(t) - \theta(t-1)\|_{\text{reg}}, \tag{3}$$

There are two key reasons why this parameter fitting problem is attractive:

- It is a convex optimization problem, so the global solution can be computed efficiently.
- The sum-of-norms form of the regularization favors solutions where "many" (depending on λ) of the regularized variables come out as exactly zero in the solution. In this case, this means estimated parameters that change infrequently (with the frequency of changes controlled roughly by λ).

In a statistical linear regression framework, sum-of-norms regularization is called Group-Lasso [3]. One final step is also useful. From our final estimate of $\theta(t)$, we simply use the set of times at which a model change occurs (i.e., for which $\theta(t) - \theta(t-1)$ is nonzero), and carry out a final least-squares fit over the parameters, which we now require to be piecewise constant over the fixed intervals. For further details, see [2].

Ex: Changing Time Delay

Consider the system (`iddemo11` in the System Identification Toolbox, [1])

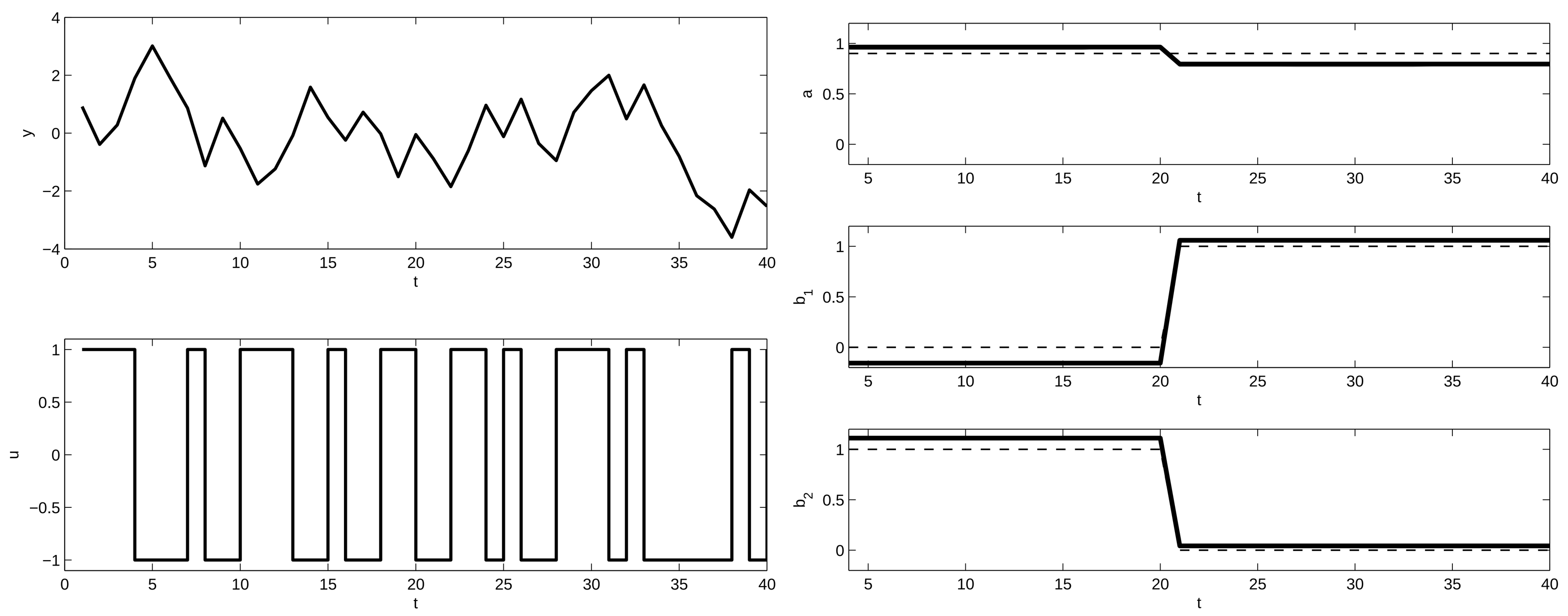
$$y(t) + 0.9y(t-1) = u(t-n_k) + e(t).$$

The input u is a ± 1 PRBS signal and the additive noise has variance 0.1. At time $t = 20$ the time delay n_k changes from 2 to 1.

An ARX-model

$$y(t) + ay(t-1) = b_1u(t-1) + b_2u(t-2)$$

is used to estimate a, b_1, b_2. The data and the resulting estimates are shown in the figure below. We clearly see that b_1 jumps from 0 to 1, to "take over" to be the leading term around sample 20.



To the left, the data used. To the right, true parameters (dashed) and parameter estimates (solid).

Extensions

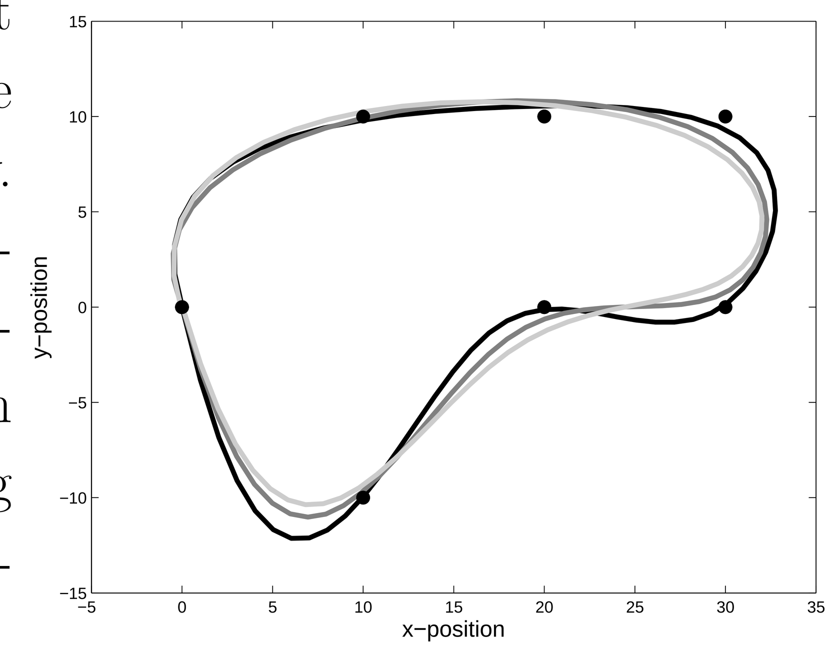
Sum-of-norms regularization can be used in many other control-related applications. Two interesting directions that we been looking in are:

Load disturbances

A *load disturbance* e.g. a step change in moment load in an electric motor, a (up or down) hill for a vehicle, etc. are not naturally modeled as Gaussian noise. This type of noise can readily be treated by adding a sum-of-norms regularization on an estimate of the process noise in a Kalman smoother.

Trajectory generation

In many tracking applications, it is of interest to generate a feasible and smooth reference trajectory. We propose a model-based optimization formulation which utilizes sum-of-norms regularization to obtain a feasible spline passing through some pre-specified way-points.



Summary

We have studied the model segmentation problem and suggested to treat it as least-squares problem with sum-of-norms regularization of the parameter changes. We do not claim that the suggested method necessarily outperforms existing approaches; but being a global method, it certainly has an edge in cases with considerable noise and infrequent jumps. An important benefit is also that it has just one scalar design variable, whose influence on the parameter fit and number of segments is easily understood.

Acknowledgment

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References

[1] L. Ljung. *The System Identification Toolbox: The Manual*. The MathWorks Inc. 1st edition 1986, 7th edition 2007, Natick, MA, USA, 2007.

[2] H. Ohlsson, L. Ljung, and S. Boyd. Segmentation of ARX-models using sum-of-norms regularization. *Automatica*, 2010. To appear.

[3] M. Yuan and Y. Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society, Series B*, 68:49–67, 2006.